A weighted Sobolev space theory of parabolic stochastic PDEs on non-smooth domains

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Abstract

In this paper we study parabolic stochastic partial differential equations (see equation (1.1)) defined on arbitrary bounded domain $\mathcal{O} \subset \mathbb{R}^d$ allowing Hardy inequality:

$$\int_{\mathcal{O}} |\rho^{-1} g|^2 \, dx \leq C \int_{\mathcal{O}} |g_x|^2 dx, \quad \forall g \in C_0^\infty(\mathcal{O}), \tag{0.1}$$

where $\rho(x) = \text{dist}(x, \partial\mathcal{O})$. Existence and uniqueness results are given in weighted Sobolev spaces $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O}, T)$, where $p \in [2, \infty)$, $\gamma \in \mathbb{R}$ is the number of derivatives of solutions and θ controls the boundary behavior of solutions (see Definition 2.5). Furthermore several Hölder estimates of the solutions are also obtained. It is allowed that the coefficients of the equations blow up near the boundary.

Keywords: Hardy inequality, Stochastic partial differential equation, non-smooth domain, L^p-theory, weighted Sobolve space.

AMS 2000 subject classifications: 60H15, 35R60.

1 Introduction

It is a classical result that Hardy inequality holds on Lipschitz domains ([31]). There have been many other works concerning Hardy inequality. See e.g. [3], [35] and references therein. We only mention that inequality (0.1) holds under much weaker condition than Lipschitz condition. For instance, it holds if $\mathcal{O}$ has plump complement, that is, there exist $b, \sigma \in (0, 1]$ such that for any $s \in (0, \sigma]$ and $x \in \partial\mathcal{O}$ there exists a point $y \in B_s(x) \cap \mathcal{O}^c$ with $\text{dist}(y, \partial\mathcal{O}) \geq bs$. For instance, $\mathcal{O}_\alpha := \{(x, y) \in \mathbb{R}^2 : x \in (-1, 1),\ |x|^\alpha + |y|^\alpha < 1\}$, where $\alpha \in (0, 1)$, is a non-Lipschitz domain but satisfies the plump complement condition.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$, each of which contains all $(\mathcal{F}, P)$-null sets. We assume that on Ω we are given independent

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one-dimensional Wiener processes $w_t^1, w_t^2, ...$ relative to $\{\mathcal{F}_t, t \geq 0\}$. The main goal of this article is to present an L_p-theory of stochastic partial differential equation

$$du = (a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f)\, dt + (\sigma^{ik} u_{x^i} + \mu^k u + g^k)\, dw_t^k \tag{1.1}$$

given for $t > 0$ and $x \in \mathcal{O}$. Here i and j go from 1 to d, and k runs through $\{1, 2, ...\}$ with the summation convention on i, j, k being enforced. The coefficients $a^{ij}, b^i, c, \sigma^{ik}, \mu^k$ and the free terms f, g^k are random functions depending on t and x. As mentioned in [20], such equations with a finite number of the processes w_t^k appear, for instance, in nonlinear filtering problems (estimations of the signal by observing it when it is mixed with noises), and considering infinitely many w_t^k is instrumental in treating equations for measure-valued processes, for instance, driven by space-time white noise (cf. [16]).

Equation (1.1) has been extensively studied by so many authors (see e.g. [4, 9, 10, 12, 14, 16, 19, 20, 28, 29, 30, 32, 36] and references therein). We give a very brief review only on the L_p-theory of the equation. The L_p-theory ($p \geq 2$) of equation (1.1) defined in $\mathbb{R}^d$ was introduced by Krylov ([16], [19]), and later Krylov and Lototsky ([20],[21]) developed a weighted L_p-theory of the equation defined on a half space. It turned out that for SPDEs defined on domains the Hölder space approach does not allow one to obtain results of reasonable generality, and the Sobolev spaces without weights are trivially inappropriate. Recently, these weighted L_p-theory on half space were extended to equations on smooth domains (e.g. [9, 10, 12, 11, 27]) and on (non-smooth) Lipschitz domain ([8]).

On non-smooth domains the spatial derivatives of the solution usually have additional singularities at the boundary which are due to the shape of the domain, see e.g. [6, 7] for the case of deterministic equations on polygonal domains and [25] for a generalization to the stochastic setting. In the context of numerical approximation this suggests the use of non-uniform schemes. In [1] results of [8] are used to prove that the convergence rates of adequate non-uniform discretization schemes are closely connected to the regularity of the solution measured in weighted Sobolev spaces.

However, we acknowledge that there is a gap in the proof of Lemma 3.1 of [8], and the main results of [8] are false unless stronger assumption on the range of weights is assumed. We show this with a counterexample. In this article we reconstruct the results in [8] under much weaker assumption on $\partial\mathcal{O}$, but with smaller range of weights. The arguments used in this article are slightly different from those in [8]. For instance, we do not use any argument of flattening the boundary, which is a key tool in [8]. Most of our important steps are based just on the Hardy inequality and Iô's formula.

As in [8, 9, 10, 12, 11, 21, 20, 27] we prove the existence and uniqueness results in weighted Sobolev classes $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O}, T)$, where $\gamma \in \mathbb{R}$ is the number of derivatives of solutions and θ controls the boundary behavior of solutions (see Definition 2.5). Also several (interior) Hölder estimates of the solutions are also obtained (see Corollary 2.14).

As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, ..., x^d)$, $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^1 > 0\}$ and $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$. For $i = 1, ..., d$, multi-indices $\beta = (\beta_1, ..., \beta_d)$, $\beta_i \in \{0, 1, 2, ...\}$, and functions $u(x)$ we set

$$u_{x^i} = \partial u / \partial x^i = D_i u, \quad D^\beta u = D_1^{\beta_1} \cdot ... \cdot D_d^{\beta_d} u, \quad |\beta| = \beta_1 + ... + \beta_d.$$

We also use the notation D^m for a partial derivative of order m with respect to x. If we write $N = N(...)$, this means that the constant N depends only on what are in parenthesis. Throughout the article, for functions depending on ω, t and x, the argument $\omega \in \Omega$ will be omitted.

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2 Main results

First we introduce some Sobolev spaces (see e.g [16], [18] and [27] for more details). Let $p \in (1, \infty)$, $\gamma \in \mathbb{R}$ and $H^\gamma_p = H^\gamma_p(\mathbb{R}^d) = (1-\Delta)^{-\gamma/2} L_p$ be the set of all distributions u such that $(1-\Delta)^{\gamma/2} u \in L_p$. Define

$$\|u\|_{H^\gamma_p} = \|(1-\Delta)^{\gamma/2} u\|_{L_p} := \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\gamma/2} \mathcal{F}(u)(\xi)]\|_p,$$

where $\mathcal{F}$ is the Fourier transform. It is well known that if γ is a nonnegative integer then

$$H^\gamma_p = H^\gamma_p(\mathbb{R}^d) = \{u : u, Du, ..., D^\gamma u \in L_p\}.$$

Denote $\rho(x) = \text{dist}(x, \partial\mathcal{O})$ and fix a bounded infinitely differentiable function ψ defined in $\mathcal{O}$ such that (see e.g. Lemma 4.13 in [22] or formula (2.6) in [26])

$$N^{-1}\rho(x) \leq \psi(x) \leq N\rho(x), \quad \rho^m |D^m \psi_x| \leq N(m) < \infty. \tag{2.1}$$

Let $\zeta \in C^\infty_0(\mathbb{R}_+)$ be a nonnegative function satisfying

$$\sum_{n=-\infty}^{\infty} \zeta(e^{n+t}) > c > 0, \quad \forall t \in \mathbb{R}. \tag{2.2}$$

Note that any non-negative smooth function $\zeta \in C^\infty_0(\mathbb{R}_+)$ so that $\zeta > 0$ on $[e^{-1}, e]$ satisfies (2.2). For $x \in \mathcal{O}$ and $n \in \mathbb{Z} := \{0, \pm 1, ...\}$ define

$$\zeta_n(x) = \zeta(e^n \psi(x)).$$

Then $\operatorname{supp} \zeta_n \subset \{x \in \mathcal{O} : e^{-n-k_0} < \rho(x) < e^{-n+k_0}\} =: G_n$ for some integer $k_0 > 0$,

$$\sum_{n=-\infty}^{\infty} \zeta_n(x) \geq \delta > 0, \tag{2.3}$$

$$\zeta_n \in C^\infty_0(G_n), \quad |D^m \zeta_n(x)| \leq N(\zeta, m) e^{mn}. \tag{2.4}$$

For $p \geq 1$ and $\gamma \in \mathbb{R}$, by $H^\gamma_{p,\theta}(\mathcal{O})$ we denote the set of all distributions u on $\mathcal{O}$ such that

$$\|u\|^p_{H^\gamma_{p,\theta}(\mathcal{O})} := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta_{-n}(e^n \cdot) u(e^n \cdot)\|^p_{H^\gamma_p} < \infty. \tag{2.5}$$

We also use the above notation for ℓ_2-valued functions $g=(g_1,g_2,...)$, that is,

$$\|g\|_{H^\gamma_p}=\|g\|_{H^\gamma_p(\ell_2)}=\||(1-\Delta)^{\gamma/2}g|_{\ell_2}\|_{L_p},$$

$$\|g\|^p_{H^\gamma_p(\mathcal{O},\ell_2)}=\sum_{n\in\mathbb{Z}}e^{n\theta}\|\zeta_{-n}(e^n\cdot)g(e^n\cdot)\|^p_{H^\gamma_p(\ell_2)}.$$

It is known (see Lemma 2.4) that if $\{\bar{\zeta}_n,n\in\mathbb{Z}\}$ is another set of functions satisfying (2.3) and (2.4) (such functions can be easily constructed by mollifying the indicator functions I_{G_n}), then it yields the same space $H^\gamma_{p,\theta}(\mathcal{O})$. Also if $\gamma=n$ is a nonnegative integer then

$$L_{p,\theta}(\mathcal{O}):=H^0_{p,\theta}(\mathcal{O})=L_p(\mathcal{O},\rho^{\theta-d}dx),$$

$$H^n_{p,\theta}(\mathcal{O}):=\{u:u,\rho Du,...,\rho^nD^nu\in L_{p,\theta}(\mathcal{O})\},$$

$$\|u\|^p_{H^n_{p,\theta}(\mathcal{O})}\sim\sum_{|\alpha|\le n}\int_{\mathcal{O}}|\rho^{|\alpha|}D^\alpha u|^p\rho^{\theta-d}\,dx. \tag{2.6}$$

We remark that the space $H^n_{p,\theta}(\mathcal{O})$ is different from $W^{n,p}(\mathcal{O},\rho,\varepsilon)$ introduced in [22], where

$$W^{n,p}(\mathcal{O},\rho,\varepsilon)=\{u:u,Du,...,D^nu\in L_p(\mathcal{O},\rho^\varepsilon dx)\}.$$

Denote $\rho(x,y)=\rho(x)\wedge\rho(y)$. For $\nu\in(0,1]$ and $k=0,1,2,...$, as in [5], define

$$[f]^{(0)}_k=[f]^{(0)}_{k,\mathcal{O}}=\sup_{\substack{x\in\mathcal{O}\\|\beta|=k}}\rho^k(x)|D^\beta f(x)|,\qquad [f]^{(0)}_{k+\nu}=\sup_{\substack{x,y\in\mathcal{O}\\|\beta|=k}}\rho^{k+\nu}(x,y)\frac{|D^\beta f(x)-D^\beta f(y)|}{|x-y|^\nu},$$

$$|f|^{(0)}_k=\sum_{j=0}^k[f]^{(0)}_{j,\mathcal{O}},\qquad |f|^{(0)}_{k+\nu}=|f|^{(0)}_k+[f]^{(0)}_{k+\nu}.$$

The above notation is used also for ℓ_2 valued functions $g=(g^1,g^2,\cdots)$. For instance,

$$[g]^{(0)}_k=\sup_{\substack{x\in\mathcal{O}\\|\beta|=k}}\rho^k(x)|D^\beta g(x)|_{\ell_2}.$$

Here are some other properties of the space $H^\gamma_{p,\theta}(\mathcal{O})$ taken from [27] (also see [17], [18]).

Lemma 2.1 (i) *The space $C^\infty_0(\mathcal{O})$ is dense in $H^\gamma_{p,\theta}(\mathcal{O})$.*

(ii) *Assume that $\gamma-d/p=m+\nu$ for some $m=0,1,...$ and $\nu\in(0,1]$, and i,j are multi-indices such that $|i|\le m,|j|=m$. Then for any $u\in H^\gamma_{p,\theta}(\mathcal{O})$, we have*

$$\psi^{|i|+\theta/p}D^iu\in C(\mathcal{O}),\quad \psi^{m+\nu+\theta/p}D^ju\in C^\nu(\mathcal{O}),$$

$$|\psi^{|i|+\theta/p}D^iu|_{C(\mathcal{O})}+[\psi^{m+\nu+\theta/p}D^ju]_{C^\nu(\mathcal{O})}\le C\|u\|_{H^\gamma_{p,\theta}(\mathcal{O})}.$$

(iii) *$\psi D,D\psi:H^\gamma_{p,\theta}(\mathcal{O})\to H^{\gamma-1}_{p,\theta}(\mathcal{O})$ are bounded linear operators, and for any $u\in H^\gamma_{p,\theta}(\mathcal{O})$*

$$\|u\|_{H^\gamma_{p,\theta}(\mathcal{O})}\le N\|\psi u_x\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})}+N\|u\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})}\le N\|u\|_{H^\gamma_{p,\theta}(\mathcal{O})},$$

$$\|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} \le N\|(\psi u)_x\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})} + N\|u\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})} \le N\|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})}.$$

(iv) For any $\nu, \gamma \in \mathbb{R}$, $\psi^{\nu} H^{\gamma}_{p,\theta}(\mathcal{O}) = H^{\gamma}_{p,\theta-p\nu}(\mathcal{O})$ *and*

$$\|u\|_{H^{\gamma}_{p,\theta-p\nu}(\mathcal{O})} \le N\|\psi^{-\nu}u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} \le N\|u\|_{H^{\gamma}_{p,\theta-p\nu}(\mathcal{O})}. \tag{2.7}$$

(v) If $\gamma \in (\gamma_0, \gamma_1)$ *and* $\theta \in (\theta_0, \theta_1)$, *then*

$$\|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} \le \varepsilon\|u\|_{H^{\gamma_1}_{p,\theta}(\mathcal{O})} + N(\gamma, p, \varepsilon)\|u\|_{H^{\gamma_0}_{p,\theta}(\mathcal{O})},$$

$$\|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} \le \varepsilon\|u\|_{H^{\gamma}_{p,\theta_0}(\mathcal{O})} + N(\gamma, p, \varepsilon)\|u\|_{H^{\gamma}_{p,\theta_1}(\mathcal{O})}.$$

Lemma 2.2 *(i) Let* $s = |\gamma|$ *if* γ *is an integer, and* $s > |\gamma|$ *otherwise, then*

$$\|au\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} \le N(d, s, \gamma)|a|^{(0)}_s \|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})}.$$

(ii) If $\gamma = 0, 1, 2, ...,$ *then*

$$\|au\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} \le N \sup_{\mathcal{O}} |a| \|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} + N_0 |a|^{(0)}_{\gamma} \|u\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})}$$

where $N_0 = 0$ *if* $\gamma = 0$.

(iii) If $0 \le r \le s$, *then*

$$|a|^{(0)}_r \le N(d, r, s)(\sup_{\mathcal{O}} |a|)^{1-r/s}(|a|^{(0)}_s)^{r/s}.$$

The assertions also holds for ℓ_2*-valued functions* a.

Proof. For (i), see Theorem 3.1 in [27]. (ii) is an easy consequence of (2.6), and (iii) is from Proposition 4.2 in [24]. □

Remark 2.3 *By Lemma 2.2, for any* $\nu \ge 0$, ψ^{ν} *is a point-wise multiplier in* $H^{\gamma}_{p,\theta}(\mathcal{O})$. *Thus if* $\theta_1 \le \theta_2$ *then*

$$\|u\|_{H^{\gamma}_{p,\theta_2}(\mathcal{O})} \le N\|\psi^{(\theta_2-\theta_1)/p}u\|_{H^{\gamma}_{p,\theta_1}(\mathcal{O})} \le N\|u\|_{H^{\gamma}_{p,\theta_1}(\mathcal{O})}.$$

Lemma 2.4 *Let* $\{\xi_n\}$ *be a sequence of* $C^{\infty}_0(\mathcal{O})$ *functions such that*

$$|D^m \xi_n| \le C(m)e^{nm}, \quad supp\, \xi_n \subset \{x \in \mathcal{O} : e^{-n-k_0} < \rho(x) < e^{-n+k_0}\}$$

for some $k_0 > 0$. *Then for any* $u \in H^{\gamma}_{p,\theta}(\mathcal{O})$,

$$\sum_n e^{n\theta} \|\xi_{-n}(e^n x)u(e^n x)\|^p_{H^{\gamma}_p} \le N\|u\|^p_{H^{\gamma}_{p,\theta}(\mathcal{O})}.$$

If in addition

$$\sum_n |\xi_n|^p > \delta > 0,$$

then the reverse inequality also holds.

Proof. See Theorem 2.2 in [27]. □

Let $\mathcal{P}$ be the predictable σ-field generated by $\{\mathcal{F}_t, t \geq 0\}$. Define

$$\mathbb{H}^\gamma_p(T) = L_p(\Omega \times [0,T], \mathcal{P}, H^\gamma_p), \quad \mathbb{H}^\gamma_p(T, \ell_2) = L_p(\Omega \times [0,T], \mathcal{P}, H^\gamma_p(\ell_2))$$

$$\mathbb{H}^\gamma_{p,\theta}(\mathcal{O}, T) = L_p(\Omega \times [0,T], \mathcal{P}, H^\gamma_{p,\theta}(\mathcal{O})), \quad \mathbb{H}^\gamma_{p,\theta}(\mathcal{O}, T, \ell_2) = L_p(\Omega \times [0,T], \mathcal{P}, H^\gamma_{p,\theta}(\mathcal{O}, \ell_2)),$$

$$\mathbb{L}_{p,\theta}(\mathcal{O}, T) = \mathbb{H}^0_{p,\theta}(\mathcal{O}, T), \quad U^\gamma_p = L_p(\Omega, \mathcal{F}_0, H^{\gamma-2/p}_p), \quad U^\gamma_{p,\theta}(\mathcal{O}) = \psi^{-\frac{2}{p}+1} L_p(\Omega, \mathcal{F}_0, H^{\gamma-2/p}_{p,\theta}(\mathcal{O})).$$

That is, for instance, we say $u \in \mathbb{H}^\gamma_{p,\theta}(\mathcal{O}, T)$ if u has a $H^\gamma_{p,\theta}(\mathcal{O})$-valued predictable version $\bar{u}$ defined on $\Omega \times [0,T]$ so that

$$\|u\|_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T)} = \|\bar{u}\|_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T)} := \left(\mathbb{E} \int_0^T \|u(s,\cdot)\|^p_{H^\gamma_{p,\theta}(\mathcal{O})} dt\right)^{1/p} < \infty.$$

Also by $u \in \psi^{-\frac{2}{p}+1} L_p(\Omega, \mathcal{F}_0, H^{\gamma-2/p}_{p,\theta}(\mathcal{O}))$ we mean $\psi^{2/p-1} u \in L_p(\Omega, \mathcal{F}_0, H^{\gamma-2/p}_{p,\theta}(\mathcal{O}))$, and

$$\|u\|^p_{U^\gamma_{p,\theta}(\mathcal{O})} := \mathbb{E}\|\psi^{2/p-1} u\|^p_{H^{\gamma-2/p}_{p,\theta}(\mathcal{O})}.$$

Below by (u, ϕ) we denote the image of $\phi \in C^\infty_0(\mathcal{O})$ under a distribution u.

Definition 2.5 *We write $u \in \mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O}, T)$ if $u \in \psi \mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O}, T)$, $u(0,\cdot) \in U^{\gamma+2}_{p,\theta}(\mathcal{O})$ and for some $f \in \psi^{-1} \mathbb{H}^\gamma_{p,\theta}(\mathcal{O}, T)$ and $g \in \mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O}, T, \ell_2)$,*

$$du = f\, dt + g^k\, dw^k_t, \tag{2.8}$$

in the sense of distributions. In other words, for any $\phi \in C^\infty_0(\mathcal{O})$, the equality

$$(u(t,\cdot), \phi) = (u(0,\cdot), \phi) + \int_0^t (f(s,\cdot), \phi)\, ds + \sum_{k=1}^\infty \int_0^t (g^k(s,\cdot), \phi)\, dw^k_s$$

holds for all $t \leq T$ with probability 1. In this situation we write $f = \mathbb{D}u$ and $g = \mathbb{S}u$. The norm in $\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O}, T)$ is defined by

$$\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} = \|\psi^{-1} u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} + \|\psi \mathbb{D}u\|_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T)} + \|\mathbb{S}u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)} + \|u(0,\cdot)\|_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}.$$

Remark 2.6 *(i) Remember that for any $\alpha, \gamma \in \mathbb{R}$, $\|\psi^\alpha u\|_{H^\gamma_{p,\theta}(\mathcal{O})} \sim \|u\|_{H^\gamma_{p,\theta+p\alpha}(\mathcal{O})}$. Thus the space $\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O}, T)$ is independent of the choice of ψ.*

(ii) It is easy to check (see Remark 3.2 of [16] for details) that for any $\phi \in C^\infty_0(\mathcal{O})$ and $g \in \mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O}, T, \ell_2)$ we have $\sum_{k=1}^\infty \int_0^T (g^k, \phi)^2 ds < \infty$, and therefore the series of stochastic integral $\sum_{k=1}^\infty \int_0^t (g^k, \phi) dw^k_t$ converges in probability uniformly on $[0,T]$.

Theorem 2.7 *Let $u_n \in \mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T), n=1,2,\cdots$ and $\|u_n\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} \le K$, where K is a finite constant. Then there exists a subsequence n_k and a function $u \in \mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$ so that*

(i) $u_{n_k}, u_{n_k}(0,\cdot), \mathbb{D}u_{n_k}, \mathbb{S}u_{n_k}$ converges weakly to $u, u(0,\cdot), \mathbb{D}u$ and $\mathbb{S}u$ in $\mathbb{H}^{\gamma+2}_{p,\theta}(T,\mathcal{O}), U^{\gamma+2}_{p,\theta}(\mathcal{O})$, $\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O})$ and $\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},\ell_2)$ respectively;

(ii) for any $\phi \in C_0^\infty(\mathcal{O})$ and $t \in [0,T]$, we have $(u_{n_k}(t,\cdot),\phi) \to (u(t,\cdot),\phi)$ weakly in $L_p(\Omega)$.

Proof. The proof is identical to that of the proof of Theorem 3.11 in [16], where the theorem is proved when $\mathcal{O}=\mathbb{R}^d$.

□

Theorem 2.8 *For any nonnegative integer $n \ge \gamma+2$, the set*

$$\mathfrak{H}^{n}_{p,\theta}(\mathcal{O},T) \bigcap \bigcup_{k=1}^{\infty} L_p(\Omega, C([0,T], C_0^n(\mathcal{O}_k))),$$

where $\mathcal{O}_k := \{x \in \mathcal{O} : \psi(x) > 1/k\}$, is dense in $\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$.

Proof. It is enough to repeat the proof of Theorem 2.9 in [20], where the lemma is proved when $\mathcal{O}=\mathbb{R}^d_+$. □

Theorem 2.9 *(i) Let $2/p < \alpha < \beta \le 1$ and $u \in \mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$, then*

$$\mathbb{E}[\psi^{\beta-1}u]^p_{C^{\alpha/2-1/p}([0,T],H^{\gamma+2-\beta}_{p,\theta}(\mathcal{O}))} \le NT^{(\beta-\alpha)p/2}\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)},$$

where N is independent of T and u.

(ii) Let $p \in [2,\infty)$ and $T<\infty$, then

$$\mathbb{E}\sup_{t\le T}\|u(t)\|^p_{H^{\gamma+1}_{p,\theta}(\mathcal{O})} \le N\|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}, \tag{2.9}$$

where $N=N(d,p,\gamma,\theta,\mathcal{O},T)$ is non-decreasing function of T. In particular, for any $t \le T$,

$$\|u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t)} \le N\int_0^t \|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},s)}\,ds. \tag{2.10}$$

Proof. The theorem is proved in [8] on Lipschitz domains, and the proof works on any arbitrary domains.

(i). Due to the definition of $\mathbb{E}[\psi^{\beta-1}u]^p_{C^{\alpha/2-1/p}([0,T],H^{\gamma+2-\beta}_{p,\theta}(\mathcal{O}))}$ we may assume $u(0)=0$. Let $\mathbb{D}u=f$ and $\mathbb{S}u=g$. By (2.5) and Lemma 2.1(iv),

$$I := \mathbb{E}[\psi^{\beta-1}u]^p_{C^{\alpha/2-1/p}([0,T],H^{\gamma+2-\beta}_{p,\theta}(\mathcal{O}))}$$

$$\le N\sum_n e^{n(\theta+p(\beta-1))}\mathbb{E}[u(t,e^n x)\zeta_{-n}(e^n x)]^p_{C^{\alpha/2-1/p}([0,T],H^{\gamma+2-\beta}_p)}. \tag{2.11}$$

Denote $T_0 := T^{(\beta-\alpha)p/2}$. By Corollary 4.12 in [15], there exists a constant $N>0$, independent of T and u, so that for any $a>0$,

$$\mathbb{E}[u(t,e^nx)\zeta_{-n}(e^nx)]^p_{C^{\alpha/2-1/p}([0,T],H^{\gamma+2-\beta}_p)} \leq NT_0a^{\beta-1}(a\|u(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma+2}_p(T)}$$
$$+a^{-1}\|f(t,e^n)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma}_p(T)}+\|g(t,e^n)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma+1}_p(T,\ell_2)}).$$

Take $a=e^{-np}$, then (2.11) yields

$$\begin{aligned} I &\leq NT_0(\sum_n e^{n(\theta-p)}\|u(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma+2}_p(T)}+\sum_n e^{n(\theta+p)}\|f(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma}_p(T)} \\ &+ \sum_n e^{n\theta}\|g(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma+1}_p(T,\ell_2)}) \\ &= NT_0\left(\|u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta-p}(\mathcal{O},T)}+\|f\|^p_{\mathbb{H}^{\gamma}_{p,\theta+p}(\mathcal{O},T)}+\|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)}\right) \leq NT_0\|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}. \end{aligned}$$

Thus (i) is proved.

(ii). If $p>2$, (ii) follows from (i). But for the case $p=2$, we prove this differently. Obviously

$$\mathbb{E}\sup_{t\leq T}\|u(t)\|^p_{H^{\gamma+1}_{p,\theta}(\mathcal{O})} \leq N\sum_n e^{n\theta}\mathbb{E}\sup_{t\leq T}\|u(t,e^nx)\zeta_{-n}(e^nx)\|^p_{H^{\gamma+1}_p}.$$

Note that $u_0\in U^{\gamma+2}_{p,\theta}(\mathcal{O})\subset L_p(\Omega,H^{\gamma+1}_{p,\theta}(\mathcal{O}))$ since $p\geq 2$. By Remark 4.14 in [15] with $\beta=1$ there, for any $a>0$,

$$\mathbb{E}\sup_{t\leq T}\|u(t,e^nx)\zeta_{-n}(e^nx)\|^p_{H^{\gamma+1}_p} \leq N(a\|u(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma+2}_p(T)}$$
$$+a^{-1}\|f(t,e^n)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma}_p(T)}+\|g(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma+1}_p(T,\ell_2)}++\mathbb{E}\|u_0(e^nx)\zeta_{-n}(e^nx)\|^p_{H^{\gamma+1}_p}).$$

Take $a=e^{-np}$ to get

$$\begin{aligned} \mathbb{E}\sup_{t\leq T}\|u(t)\|^p_{H^{\gamma+1}_{p,\theta}(\mathcal{O})} &\leq N(\sum_n e^{n(\theta-p)}\|u(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma+2}_p(T)}+\sum_n e^{n(\theta+p)}\|f(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma}_p(T)} \\ &+\sum_n e^{n\theta}\|g(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma+1}_p(T)}+\mathbb{E}\sum_n e^{n\theta}\|u_0(e^nx)\zeta_{-n}(e^nx)\|^p_{H^{\gamma+1}_p}) \\ &= N(\|u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta-p}(\mathcal{O},T)}+\|f\|^p_{\mathbb{H}^{\gamma}_{p,\theta+p}(\mathcal{O},T)}+\|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O})}+\mathbb{E}\|u_0\|^p_{H^{\gamma+1}_{p,\theta}(\mathcal{O})}) \\ &\leq N\|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}. \end{aligned}$$

Finally,

$$\|u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t)}=\mathbb{E}\int_0^t\|u\|^p_{H^{\gamma+1}_{p,\theta}(\mathcal{O})}ds\leq\int_0^t(\mathbb{E}\sup_{r\leq s}\|u(r)\|^p_{H^{\gamma+1}_{p,\theta}(\mathcal{O})})ds\leq N\int_0^t\|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},s)}ds.$$

The theorem is proved. □

Fix a nonnegative constant $\varepsilon_0=\varepsilon(\gamma)\geq 0$ so that $\varepsilon_0>0$ only if γ is not integer, and define $\gamma_+=|\gamma|$ if γ is an integer, and $\gamma_+=|\gamma|+\varepsilon_0$ otherwise. Now we state our assumptions on the coefficients.

Assumption 2.10 *(i) For each x, the coefficients $a^{ij}(t,x)$, $b^i(t,x)$ $c(t,x)$, $\sigma^{ik}(t,x)$ and $\mu^k(t,x)$ are predictable functions of (ω,t).*

(ii) The coefficients a^{ij},σ^i are uniformly continuous in x, that is, for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that

$$|a^{ij}(t,x)-a^{ij}(t,y)|+|\sigma^i(t,x)-\sigma^i(t,y)|_{\ell_2}\le\varepsilon$$

for each ω,t, whenever $x,y\in\mathcal{O}$ and $|x-y|\le\delta$.

(iii) There exist constant $\delta_0,K>0$ such that for any ω,t,x and $\lambda\in\mathbb{R}^d$,

$$\delta_0|\lambda|^2\le\bar{a}^{ij}(t,x)\lambda^i\lambda^j\le K|\lambda|^2, \tag{2.12}$$

where $\bar{a}^{ij}=a^{ij}-\frac{1}{2}(\sigma^i,\sigma^j)_{\ell_2}$.

(iv) For any ω,t

$$|a^{ij}(t,\cdot)|^{(0)}_{\gamma_+}+|\psi b^i(t,\cdot)|^{(0)}_{\gamma_+}+|\psi^2c(t,\cdot)|^{(0)}_{\gamma_+}+|\sigma^i(t,\cdot)|^{(0)}_{(\gamma+1)_+}+|\psi\mu(t,\cdot)|^{(0)}_{(\gamma+1)_+}\le K, \tag{2.13}$$

and if $\gamma=0$, then for some $\varepsilon>0$,

$$|\sigma^i(t,\cdot)|^{(0)}_{1+\varepsilon}+|\psi\mu|^{(0)}_{1+\varepsilon}\le K,\qquad\forall\omega,t. \tag{2.14}$$

(v) There is a control on the behavior of b^i,c and μ^k near $\partial\mathcal{O}$, namely,

$$\lim_{\rho(x)\to0}\sup_{\omega,t}\left(\rho(x)|b^i(t,x)|+\rho^2(x)|c(t,x)|+\rho(x)|\mu(t,x)|_{\ell_2}\right)=0. \tag{2.15}$$

Remark 2.11 *Conditions (2.13) and (2.15) allow the coefficients b^i,c and ν to be unbounded and to blow up near the boundary. In particular, (2.15) is satisfied if for some $\varepsilon,N>0$,*

$$|b^i(t,x)|+|\mu(t,x)|_{\ell_2}\le N\rho^{-1+\varepsilon}(x),\quad |c(t,x)|\le N\rho^{-2+\varepsilon}(x).$$

The proof of following theorem is given in section 4.

Theorem 2.12 *Let $p\in[2,\infty),\gamma\in[0,\infty),T<\infty$ and Assumption 2.10 be satisfied. Then there exists $\beta_0=\beta_0(p,d,\mathcal{O})>0$ so that if*

$$\theta\in(p-2+d-\beta_0,p-2+d+\beta_0) \tag{2.16}$$

then for any $f\in\psi^{-1}\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)$, $g\in\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)$ and $u_0\in U^{\gamma+2}_{p,\theta}(\mathcal{O})$ equation (1.1) with initial data u_0 admits a unique solution u in the class $\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$, and for this solution

$$\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}\le C(\|\psi f\|_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)}+\|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)}+\|u_0\|_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}), \tag{2.17}$$

where $C=C(d,p,\gamma,\theta,\delta_0,K,T,\mathcal{O})$.

Remark 2.13 *Note that Theorem 2.12 is proved only for $\gamma \geq 0$. However the theorem can be extended for any $\gamma \in \mathbb{R}$ by using results for $\gamma \geq 0$ and arguments used e.g. in the proof of Theorem 2.16 of [12] (cf. [11, 20]). One difference is that, in place of Theorem 2.8 of [18], one has to use the corresponding version on bounded domains (Theorem 5.1 of [27]).*

Lemma 2.1(ii) and Theorem 2.9 easily yield the following result.

Corollary 2.14 *Let $u \in \mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},\tau)$ be the solution in Theorem 2.12 (or in Theorem 2.15 below).*

(i) If $\gamma + 2 - d/p = m + \nu$ for some $m = 0, 1, ...,$ $\nu \in (0, 1]$, and i, j are multi-indices such that $|i| \leq m, |j| = m$, then for each ω, t

$$\psi^{|i|-1+\theta/p} D^i u \in C(\mathcal{O}), \quad \psi^{m-1+\nu+\theta/p} D^j u \in C^{\nu}(\mathcal{O}).$$

In particular,

$$|\psi^{|i|} D^i u(x)| \leq N\psi^{1-\theta/p}(x).$$

(ii) Let

$$2/p < \alpha < \beta \leq 1, \quad \gamma + 2 - \beta - d/p = k + \varepsilon$$

where $k \in \{0, 1, 2, ...\}$ and $\varepsilon \in (0, 1]$. Denote $\delta = \beta - 1 + \theta/p$. Then for any multi-indices i and j such that $|i| \leq k$ and $|j| = k$, we have

$$\mathbb{E} \sup_{t,s \leq \tau} |t-s|^{-(p\alpha/2-1)} (|\psi^{\delta+|i|} D^i(u(t) - u(s))|^p_{C(\mathcal{O})}$$

$$+[\psi^{\delta+|j|+\varepsilon} D^j(u(t) - u(s))]^p_{C^{\varepsilon}(\mathcal{O})}) < \infty.$$

Note that if $p = 2$ then (2.16) is $\theta \in (d - \beta_0, d + \beta_0)$, but it is not clear whether d is included in the interval in (2.16) if $p > 2$ because β_0 depends also on p. Below we give positive answer if p is close to 2 and negative one for large p. For instance, **if $\theta = d = 2$ then in general Theorem 2.12 do not hold for all** $p > 4$.

The proof of following theorem is given in section 5.

Theorem 2.15 *There exists $p_0 > 2$ so that if $p \in [2, p_0)$ then there exists $\beta_1 > 0$ so that the assertion of Theorem 2.12 holds for any $\theta \in (d - \beta_1, d + \beta_1)$.*

Remark 2.16 *Since $H^1_{p,d-p}(\mathcal{O}) = \overset{\circ}{W}{}^1_p(\mathcal{O}) := \{u : u, u_x \in L_p(\mathcal{O}) \text{ and } u|_{\partial\mathcal{O}} = 0\}$, Theorem 2.15 with $\gamma \geq -1$ and $\theta \in (d - \beta_1, d]$ implies that there exists a unique solution $u \in L_p(\Omega \times [0, T], \overset{\circ}{W}{}^1_p(\mathcal{O}))$ for any $p \in [2, p_0)$.*

The following example is due to N.V. Krylov and shows that Theorem 2.12 can not hold unless θ is sufficiently large and that in general Theorem 2.15 is false for all large p.

Example 2.17 *Let $\alpha \in (1/2, 1)$ and denote*

$$G_\alpha = \{z = x + iy : |arg\, z| < \frac{\pi}{2\alpha}\}, \quad \mathcal{O}_\alpha = G_\alpha \cap \{z : |z| < 10\},$$

where $arg\, z$ is defined as a function taking values in so that $[\pi, -\pi)$. Define $v(z) = v(x, y) = Re\, z^\alpha = |z|^\alpha \cos \alpha\theta$, where $\tan\theta = y/x$. Then $\Delta v = 0$ in G_α and $v = 0$ on ∂G_α. We claim that for some $N = N(\alpha) > 1$,

$$N^{-1}|z|^{\alpha-1} \le |\rho^{-1}v| \le N|z|^{\alpha-1}, \quad |Dv| + |\rho D^2 v| \le N|z|^{\alpha-1}.$$

Since the second assertion is easy to check we prove the first one. If $|arg\, z| < \frac{\pi}{2\alpha} - \frac{\pi}{2}$ then $\rho(z) = |z|$, $|z|^\alpha \cos(\frac{\pi}{2} - \alpha\frac{\pi}{2}) \le |v| \le |z|^\alpha$ and the claim is obvious. Also if $\frac{\pi}{2\alpha} - \frac{\pi}{2} \le |arg\, z| < \frac{\pi}{2\alpha}$, then $\rho(z) = |z||\sin(\frac{\pi}{2\alpha} - \theta)|$ and $\cos\alpha\theta/|\sin(\frac{\pi}{2\alpha} - \theta|$ is comparable to 1 in $\{z : \frac{\pi}{2\alpha} - \frac{\pi}{2} \le |arg\, z| < \frac{\pi}{2\alpha}\}$.

It follows that

$$\int_{\mathcal{O}_\alpha} \left(|\rho^{-1}v|^p + |Dv| + |\rho D^2 v|\right) \rho^{\theta-2} dx < \infty \quad \Leftrightarrow \quad \theta > p(1-\alpha),$$

and

$$\int_{\mathcal{O}_\alpha} (|\rho v_x|^p + |\rho v|^p)\rho^{\theta-2} dx < \infty, \quad \forall\, \theta > 0.$$

Now choose a smooth function $\xi \in C_0^\infty(B_2(0))$ so that $\xi = 1$ on $B_1(0)$, and define $u(t, x, y) := t\xi(x, y)v(x, y)$. Then

$$du = (\Delta u + f)dt, \tag{2.18}$$

where $f := t(-2\xi_{x^i} v_{x^i} - v\Delta\xi) + \xi v$. Above calculations show that $\rho f \in \mathbb{L}_{p,\theta}(\mathcal{O}_\alpha, T)$ for any $\theta > 0$ and that $u \in \mathfrak{H}^2_{p,p}(\mathcal{O}_\alpha, T)$. By Theorem 2.12 we conclude that u is the unique solution of the above equation in $\mathfrak{H}^2_{p,p}(\mathcal{O}_\alpha, T)$. It also follows that **the existence result of Theorem 2.12 in $\mathfrak{H}^2_{p,\theta}(\mathcal{O}_\alpha, T)$ fails** *whenever*

$$\theta \le p(1-\alpha),$$

because if there is any solution $w \in \mathfrak{H}^2_{p,\theta}(\mathcal{O}_\alpha, T)$ then $w \in \mathfrak{H}^2_{p,p}(\mathcal{O}_\alpha, T)$ and therefore due to the uniqueness result in $\mathfrak{H}^2_{p,p}(\mathcal{O}_\alpha, T)$, we get $u = w$. But this is not possible since $\|\rho^{-1}u\|_{\mathbb{L}_{p,\theta}(\mathcal{O}_\alpha, T)} = \infty$. In particular, if $\theta = d = 2$ and $p > 4$ we can choose α close to $1/2$ so that $2 \le p(1-\alpha)$, and consequently this leads to the fact that in general Theorem 2.12 does not holds if $p > 4$.

3 A priori estimate

In this section we develop some estimations of solutions of equation (1.1). First, we introduce a result on SPDEs defined on entire space $\mathbb{R}^d$.

Lemma 3.1 *Let a^{ij} and σ^{ij} be independent of x. Also suppose that $f \in \mathbb{H}^\gamma_p(T)$, $g \in \mathbb{H}^{\gamma+1}_p(T, \ell_2)$, $u_0 \in U^{\gamma+2}_p$ and $u \in \mathbb{H}^{\gamma+1}_p(T)$ is a solution of*

$$du = (a^{ij}u_{x^i x^j} + f) + (\sigma^{ik}u_{x^i} + g^k)dw^k_t \quad u(0, \cdot) = u_0. \tag{3.1}$$

Then $u \in \mathbb{H}_p^{\gamma+2}(T)$*, and*

$$\|u\|^p_{\mathbb{H}_p^{\gamma+2}(T)} \leq N(\|u\|^p_{\mathbb{H}_p^{\gamma+1}(T)} + \|f\|^p_{\mathbb{H}_p^{\gamma}(T)} + \|g\|^p_{\mathbb{H}_p^{\gamma+1}(T,\ell_2)} + \|u_0\|^p_{U_p^{\gamma+2}}), \tag{3.2}$$

where N *depends only on* d, p, δ_0, K *(not on* T*).*

Proof. This is a well known result. By Theorem 4.10 in [16],

$$\|u_{xx}\|^p_{\mathbb{H}_p^{\gamma}(T)} \leq C(d,p)(\|f\|^p_{\mathbb{H}_p^{\gamma}(T)} + \|g\|^p_{\mathbb{H}_p^{\gamma+1}(T,\ell_2)} + \|u_0\|^p_{U_p^{\gamma+2}}).$$

This and the relation $\|u\|_{H_p^{\gamma+2}} = \|(1-\Delta)u\|_{H_p^{\gamma}} \leq (\|u\|_{H_p^{\gamma}} + \|u_{xx}\|_{H_p^{\gamma}})$ certainly prove (3.2). □

In the following lemma there is no restriction on θ, γ and $\partial\mathcal{O}$, that is $\theta, \gamma \in \mathbb{R}$ and $\mathcal{O}$ is any arbitrary domain.

Lemma 3.2 *Let* a^{ij} *and* σ^{ik} *be independent of* x*. Suppose* $f \in \psi^{-1}\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)$*,* $g \in \mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)$*,* $u_0 \in U^{\gamma+2}_{p,\theta}(\mathcal{O})$ *and* $u \in \mathfrak{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)$ *is a solution of the equation*

$$du = (a^{ij}u_{x^ix^j} + f)dt + (\sigma^{ik}u_{x^i} + g^k)dw^k_t, \qquad u(0,\cdot) = u_0.$$

Then $u \in \mathfrak{H}^{\gamma+2}_{p,\theta}(\Omega,T)$*, and*

$$\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} \leq N(\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)} + \|\psi f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)} + \|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}). \tag{3.3}$$

Proof. We just repeat the arguments used in [8] on Lipschitz domains. Remember that by Lemma 2.1 we have $\|\psi^{-1}u\|_{H^{\gamma+2}_{p,\theta}(\mathcal{O})} \sim \|u\|_{H^{\gamma+2}_{p,\theta-p}(\mathcal{O})}$. Thus,

$$\begin{aligned}
\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} &\leq N\sum_{n=-\infty}^{\infty} e^{n(\theta-p)}\|u(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}_p^{\gamma+2}(T)} \\
&= N\sum_{n=-\infty}^{\infty} e^{n(\theta-p+2)}\|v_n\|^p_{\mathbb{H}_p^{\gamma+2}(e^{-2n}T)},
\end{aligned} \tag{3.4}$$

where $v_n(t,x) := u(e^{2n}t, e^nx)\zeta_{-n}(e^nx)$. Note that since v_n has compact support in $\mathbb{R}^d$ and can be regarded as distribution defined on $\mathbb{R}^d$. Thus we conclude $v_n \in \mathbb{H}_p^{\gamma+1}(e^{-2n}T)$. Also note that it satisfies

$$dv_n = (a^{ij}(e^{2n}t)v_{nx^ix^j} + f_n)dt + (\sigma^{ik}(e^{2n}t)v_{nx^i} + g^k_n)dw^k(n)_t, \quad v_n(0) = u_0(e^nx)\zeta_{-n}(e^nx),$$

where $w^k(n)_t := e^{-n}w^k_{e^nt}$ are independent Wiener processes,

$$\begin{aligned}
f_n(t,x) &= -2e^na^{ij}(e^{2n}t,x)u_{x^i}(e^{2n}t,e^nx)e^n\zeta_{-nx^j}(e^nx) - a^{ij}u(e^{2n}t,e^nx)e^{2n}\zeta_{-nx^ix^j}(e^nx) \\
&+ e^{2n}f(e^{2n}t,e^nx)\zeta_{-n}(e^nx),
\end{aligned}$$

and

$$g^k_n = -\sigma^{ik}(e^{2n}t)u(e^{2n}t,e^nx)e^n\zeta_{-nx^i}(e^nx) + e^ng^k(e^{2n}t,e^nx)\zeta_{-n}(e^nx).$$

Since ζ_{-n} has compact support in $\mathcal{O}$ and $u \in \mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)$, we easily check that

$$f_n \in \mathbb{H}^{\gamma}_p(e^{-2n}T), \quad g_n \in \mathbb{H}^{\gamma+1}_p(e^{-2n}T,\ell_2).$$

Thus by Lemma 3.1, we have $v_n \in \mathbb{H}^{\gamma+2}_p(e^{2n}T)$ and

$$\|v_n\|^p_{\mathbb{H}^{\gamma+2}_p(e^{2n}T)} \leq N(\|v_n\|^p_{\mathbb{H}^{\gamma+1}_p(e^{2n}T)} + \|f_n\|^p_{\mathbb{H}^{\gamma}_p(e^{2n}T)} + \|g_n\|^p_{\mathbb{H}^{\gamma+1}_p(e^{2n}T,\ell_2)} + \|v_n(0)\|^p_{U^{\gamma+2}_p}),$$

where $N = N(d,p,\gamma,\delta_0,K)$ is independent of n and T. Next we apply Lemma 2.4 with $\xi_n = e^{-n}\zeta_{nx^i}$ or $\xi_n = e^{-2n}\zeta_{nx^ix^j}$ and get

$$\begin{aligned}\sum_{n=-\infty}^{\infty} e^{n(\theta-p+2)}\|f_n\|^p_{\mathbb{H}^{\gamma}_p(e^{-2n}T)} &\leq N\sum_n e^{n\theta}\|u_{x^i}(t,e^nx)e^n\zeta_{-nx^j}(e^nx)\|^p_{\mathbb{H}^{\gamma}_p(T)} \\ &+ N\sum_n e^{n(\theta-p)}\|u(t,e^nx)e^{2n}\zeta_{-nx^ix^j}(e^nx)\|^p_{\mathbb{H}^{\gamma}_p(T)} \\ &+ N\sum_n e^{n(\theta+p)}\|f(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma}_p(T)} \\ &\leq N\|u_x\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)} + N\|u\|^p_{\mathbb{H}^{\gamma}_{p,\theta-p}(\mathcal{O},T)} + N\|f\|^p_{\mathbb{H}^{\gamma}_{p,\theta+p}(\mathcal{O},T)} \\ &\leq N\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)} + N\|\psi f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} e^{n(\theta-p+2)}\|g_n\|^p_{\mathbb{H}^{\gamma+1}_p(e^{-2n}T)} \\ \leq\ & N\sum_n e^{n(\theta-p)}\|u(t,e^nx)e^n\zeta_{-nx}(e^nx)\|^p_{\mathbb{H}^{\gamma+1}_p(T)} + N\sum_n e^{n\theta}\|g(t,e^nx)\zeta_{-n}(e^nx)\|^p_{\mathbb{H}^{\gamma+1}_p(T,\ell_2)} \\ \leq\ & N\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)} + N\|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)}.\end{aligned}$$

Also,

$$\sum_{n=-\infty}^{\infty} e^{n(\theta-p+2)}\|v_n(0)\|^p_{U^{\gamma+2}_p} \leq N\|u_0\|^p_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}.$$

Thus the lemma is proved.

□

Remark 3.3 *Let $\gamma \geq 0$. By (3.3) and the inequality (see Lemma 2.1(v))*

$$\|\psi^{-1}u\|_{H^{\gamma+1}_{p,\theta}(\mathcal{O})} \leq \varepsilon\|\psi^{-1}u\|_{H^{\gamma+2}_{p,\theta}(\mathcal{O})} + N(\varepsilon)\|\psi^{-1}u\|_{L_{p,\theta}(\mathcal{O})},$$

we easily get

$$\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} \leq N(\|\psi^{-1}u\|^p_{\mathbb{L}_{p,\theta}(\mathcal{O},T)} + \|\psi f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)} + \|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}). \quad (3.5)$$

This shows that to estimate $\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O})}$ it is enough to estimate $\|\psi^{-1}u\|^p_{\mathbb{L}_{p,\theta}(\mathcal{O},T)}$.

In the following lemma we estimate $\|\psi^{-1}u\|^p_{\mathbb{L}_{p,\theta}(\mathcal{O},T)}$ when $\theta = d-2+p$ using the Hardy inequality.

Lemma 3.4 *Let a^{ij} and σ^{ik} be independent of x. Then for any $u \in \mathfrak{H}^2_{p,d-2+p}(\mathcal{O},T)$, we have*

$$\begin{aligned}\|u\|_{\mathfrak{H}^2_{p,d-2+p}(\mathcal{O},T)} &\leq N\|\psi(\mathbb{D}u - a^{ij}u_{x^ix^j})\|_{\mathbb{L}_{p,d-2+p}(\mathcal{O},T)} \\ &+ N\|\mathbb{S}u - \sigma^{i\cdot}u_{x^i}\|_{\mathbb{H}^1_{p,d-2+p}(\mathcal{O},T,\ell_2)} + N\|u(0)\|_{U^2_{p,d-2+p}(\mathcal{O})},\end{aligned} \tag{3.6}$$

where $N = N(d,p,\mathcal{O})$.

Proof. Step 1. First assume that $u \in L_p(\Omega, C([0,T], C^2_0(\mathcal{O}_k)))$ for some k, where $\mathcal{O}_k := \{x \in \mathcal{O} : \psi(x) > 1/k\}$, so that u is sufficiently smooth in x and vanishes near the boundary $\partial\mathcal{O}$. Denote

$$f = \mathbb{D}u - a^{ij}u_{x^ix^j}, \quad g = \mathbb{S}u - \sigma^{ik}u_{x^i}, \quad u_0 = u(0).$$

Then for each $x \in \mathcal{O}$,

$$u(t,x) = u_0(x) + \int_0^t (a^{ij}u_{x^ix^j} + f)ds + \int_0^t (\sigma^{ik}u_{x^i} + g^k)dw^k_t,$$

for all $t \leq T$ (a.s.). Applying Itô's formula to $|u(t,x)|^p$,

$$\begin{aligned}|u(T)|^p &= |u_0|^p + p\int_0^T |u|^{p-2}u(a^{ij}u_{x^ix^j} + f)\,dt + \int_0^T p|u|^{p-2}u(\sigma^{ik}u_{x^i} + g^k)dw^k_t \\ &+ \frac{1}{2}p(p-1)\int_0^T |u|^{p-2}\sum_{k=1}^\infty(\sigma^{ik}u_{x^i} + g^k)^2 dt.\end{aligned}$$

Note that

$$\frac{1}{2}p(p-1)|u|^{p-2}\sum_{k=1}^\infty(\sigma^{ik}u_{x^i} + g^k)^2 = p(p-1)|u|^{p-2}\left(\alpha^{ij}u_{x^i}u_{x^j} + u_{x^i}(\sigma^i, g)_{\ell_2} + \frac{1}{2}|g|^2_{\ell_2}\right),$$

where $\alpha^{ij} = \frac{1}{2}(\sigma^i, \sigma^j)_{\ell_2}$. Taking expectation, integrating over $\mathcal{O}$ and doing integration by parts (that is, $\int_{\mathcal{O}} p|u|^{p-2}ua^{ij}u_{x^ix^j}\,dx = -p(p-1)\int_{\mathcal{O}} a^{ij}|u|^{p-2}u_{x^i}u_{x^j}\,dx$), we get

$$\begin{aligned}p(p-1)\mathbb{E}\int_0^T\int_{\mathcal{O}} \bar{a}^{ij}|u|^{p-2}u_{x^i}u_{x^j}\,dxdt &\leq \mathbb{E}\int_{\mathcal{O}}|u_0|^p dx + p\mathbb{E}\int_0^T\int_{\mathcal{O}}|u|^{p-1}|f|dxdt \\ &+ p(p-1)\int_0^t\int_{\mathcal{O}}|u|^{p-2}(u_{x^i}(\sigma^i,g)_{\ell_2} + \frac{1}{2}|g|^2_{\ell_2})dxdt.\end{aligned}$$

Note that for each ω, t we have $v := |u|^{p/2} \in \{f : f, f_x \in L_2(\mathcal{O}), f|_{\partial\mathcal{O}} = 0\}$, and $v_x = \frac{p}{2}|u|^{p/2-2}uu_x$. Thus by Hardy Inequality (see (0.1)),

$$\int_{\mathcal{O}}|\psi^{-1}u|^p\psi^{p-2}dx = \int_{\mathcal{O}}|\psi^{-1}v|^2dx \leq N\int_{\mathcal{O}}|v_x|^2dx \leq N\int_{\mathcal{O}}|u|^{p-2}|u_x|^2dx. \tag{3.7}$$

Also note that

$$\int_{\mathcal{O}} |u|^{p-1}|f|dx = \int_{\mathcal{O}} |\psi^{-1}u|^{p-1}|\psi f|\psi^{p-2}dx \quad \leq \quad \varepsilon \int_{\mathcal{O}} |\psi^{-1}u|^p \psi^{p-2}dx + N(\varepsilon)\int_{\mathcal{O}} |\psi f|^p \psi^{p-2}dx,$$

$$\begin{aligned}
\int_{\mathcal{O}} |u|^{p-2}u_{x^i}(\sigma^i, g)_{\ell_2}dx &\leq N|\sigma|_{\ell_2}\int_{\mathcal{O}} |u|^{p-2}|u_x||g|_{\ell_2}dx \\
&= N|\sigma|_{\ell_2}\int_{\mathcal{O}} |\psi^{-1}u|^{p-2}|u_x||g|_{\ell_2}\psi^{p-2}dx \\
&\leq \varepsilon\int_{\mathcal{O}} |\psi^{-1}u|^p\psi^{p-2}dx + \varepsilon\int_{\mathcal{O}} |u_x|^p\psi^{p-2}dx + N(\varepsilon)\int_{\mathcal{O}} |g|^p_{\ell_2}\psi^{p-2}dx.
\end{aligned}$$

Similarly,

$$\int_{\mathcal{O}} |u|^{p-2}|g|^2_{\ell_2}dx \leq \varepsilon\int_{\mathcal{O}} |\psi^{-1}u|^p\psi^{p-2}dx + N(\varepsilon)\int_{\mathcal{O}} |g|^p_{\ell_2}\psi^{p-2}dx.$$

Since $(\bar{a}^{ij}) \geq \delta_0 I$, we have $\delta|u|^{p-2}|Du|^2 \leq \bar{a}^{ij}|u|^{p-2}u_{x^i}u_{x^j}$, and therefore from above calculations

$$\begin{aligned}
(1-N_0\varepsilon)\mathbb{E}\int_0^T\int_{\mathcal{O}} |\psi^{-1}u|^p\psi^{p-2}dxdt &\leq N\mathbb{E}\int_{\mathcal{O}} |\psi^{\frac{2}{p}-1}u(0)|^p\psi^{p-2}\,dx + N\varepsilon\mathbb{E}\int_0^T\int_{\mathcal{O}} |u_x|^p\psi^{p-2}\,dx\,dt \\
&+ N(\varepsilon)\mathbb{E}\int_0^T\int_{\mathcal{O}} |\psi f|^p\psi^{p-2}dxdt + N(\varepsilon)\mathbb{E}\int_0^T\int_{\mathcal{O}} |g|^p_{\ell_2}\psi^{p-2}dxdt.
\end{aligned}$$

Thus for any $\varepsilon > 0$ so that $\varepsilon N_0 < 1/2$, we have

$$\begin{aligned}
\|\psi^{-1}u\|_{\mathbb{L}_{p,d-2+p}(\mathcal{O},T)} &\leq N\|u_0\|_{U^1_{p,d-2+p}} + N\varepsilon\|u_x\|_{\mathbb{L}_{p,d-2+p}(\mathcal{O},T)} \\
&+ N(\varepsilon)\|\psi f\|_{\mathbb{L}_{p,d-2+p}(\mathcal{O},T)} + N(\varepsilon)\|g\|_{\mathbb{L}_{p,d-2+p}(\mathcal{O},T,\ell_2)}.
\end{aligned} \tag{3.8}$$

This and (3.5) easily lead to (3.6).

Step 2. General case. We use Theorem 2.8. Take a sequence $u^n \in \mathfrak{H}^2_{p,d-2+p}(\mathcal{O},T)$ so that $u^n \to u$ in $\mathfrak{H}^2_{p,d-2+p}(\mathcal{O},T)$ and each $u^n \in L_p(\Omega, C([0,T], C^2_0(G_k)))$ for some $k = k(n)$. By Step 1, we have (3.6) with u^n in place of u. Now it is enough to let $n \to \infty$. □

The following lemma virtually says that if Theorem 2.12 holds for some $\theta_0 \in \mathbb{R}$, then it also holds for all θ near θ_0.

Lemma 3.5 *Suppose that there exists a $\theta_0 \in \mathbb{R}$ so that for any $u \in \mathfrak{H}^2_{p,\theta_0}(\mathcal{O},T)$ we have*

$$\|u\|_{\mathfrak{H}^2_{p,\theta_0}(\mathcal{O},T)} \leq N\left(\|\psi\mathbb{D}u - \psi a^{ij}u_{x^ix^j}\|_{\mathbb{L}_{p,\theta_0}(\mathcal{O},T)} + \|\mathbb{S}u - \sigma^i u_{x^i}\|_{\mathbb{H}^1_{p,\theta_0}(\mathcal{O},T,\ell_2)} + \|u(0)\|_{U^2_{p,\theta_0}(\mathcal{O})}\right). \tag{3.9}$$

Then there exists $\varepsilon_0 = \varepsilon_0(N,\theta_0,p) > 0$ so that for any $\theta \in (\theta_0-\varepsilon_0, \theta_0+\varepsilon_0)$ and $v \in \mathfrak{H}^2_{p,\theta}(\mathcal{O},T)$ it holds that

$$\|v\|_{\mathfrak{H}^2_{p,\theta}(\mathcal{O},T)} \leq N\left(\|\psi\mathbb{D}v - \psi a^{ij}v_{x^ix^j}\|_{\mathbb{L}_{p,\theta}(\mathcal{O},T)} + \|\mathbb{S}v - \sigma^i v_{x^i}\|_{\mathbb{H}^1_{p,\theta}(\mathcal{O},T,\ell_2)} + \|v(0)\|_{U^2_{p,\theta}(\mathcal{O})}\right).$$

Proof. Let $v \in \mathfrak{H}^2_{p,\theta}(\mathcal{O},T)$. Denote $\nu = (\theta_0-\theta)/p$ and $u := \psi^\nu v$, then by (2.7), $u \in \mathfrak{H}^2_{p,\theta_0}(\mathcal{O},T)$. Also it is easy to check that $\mathbb{D}u = \psi^\nu \mathbb{D}v$, $\mathbb{S}u = \psi^\nu \mathbb{S}v$ and

$$\mathbb{D}u - a^{ij}u_{x^ix^j} = \psi^\nu(\mathbb{D}v - a^{ij}v_{x^ix^j}) - 2a^{ij}v_{x^i}(\psi^\nu)_{x^j} - a^{ij}v(\psi^\nu)_{x^ix^j},$$

$$\mathbb{S}u - \sigma^i u_{x^i} = \psi^\nu(\mathbb{S}v - \sigma^i v_{x^i}) - \sigma^i v(\psi^\nu)_{x^i}.$$

Note, since ψ_x and $\psi\psi_{xx}$ are bounded, if $\nu \le 1$ then

$$|(\psi^\nu)_{x^j}| = \nu|\psi^{\nu-1}\psi_{x^i}| \le N\nu\psi^{\nu-1}, \qquad |(\psi^\nu)_{x^ix^j}| \le N\nu\psi^{\nu-2}. \tag{3.10}$$

By assumption (see (3.9)) and (3.10))

$$\begin{aligned}\|\psi^\nu v\|_{\mathfrak{H}^2_{p,\theta_0}(\mathcal{O},T)} &\le N\|\psi^\nu\psi(\mathbb{D}v - a^{ij}v_{x^ix^j})\|_{\mathbb{L}_{p,\theta_0}(\mathcal{O},T)} + N\|\psi^\nu(\mathbb{S}v - \sigma^i v_{x^i})\|_{\mathbb{H}^1_{p,\theta_0}(\mathcal{O},T,\ell_2)} \\ &+ N\nu(\|\psi^\nu\psi^{-1}v\|_{\mathbb{H}^1_{p,\theta_0}(\mathcal{O},T)} + \|\psi^\nu v_x\|_{\mathbb{L}_{p,\theta_0}(\mathcal{O},T)}) + N\|\psi^\nu v(0)\|_{U^2_{p,\theta_0}(\mathcal{O})}.\end{aligned}$$

This certainly implies (see (2.7))

$$\begin{aligned}\|v\|_{\mathfrak{H}^2_{p,\theta}(\mathcal{O},T)} &\le N\|\psi(\mathbb{D}v - a^{ij}v_{x^ix^j})\|_{\mathbb{L}_{p,\theta}(\mathcal{O},T)} + N\|\mathbb{S}v - \sigma^i v_{x^i}\|_{\mathbb{H}^1_{p,\theta}(\mathcal{O},T,\ell_2)} \\ &+ N_1\nu\left(\|\psi^{-1}v\|_{\mathbb{H}^1_{p,\theta}(\mathcal{O},T)} + \|v_x\|_{\mathbb{L}_{p,\theta}(\mathcal{O},T)}\right) + N\|v(0)\|_{U^2_{p,\theta}(\mathcal{O})}.\end{aligned}$$

It follows that the claim of the lemma holds for all sufficiently small ν, that is for any θ so that $N_1|\theta_0-\theta|/p < 1$. The lemma is proved. □

Remark 3.3, Lemma 3.4 and Lemma 3.5 obviously lead to the following result.

Corollary 3.6 *Suppose that $\gamma \ge 0$ and the coefficients a^{ij}, σ^{ik} are independent of x. Then there exists $\beta_0 = \beta_0(d,p,\mathcal{O}) > 0$ so that if $\theta \in (d-2+p-\beta_0, d-2+p+\beta_0)$, $f \in \mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T)$, $g \in \mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)$, $u_0 \in U^{\gamma+2}_{p,\theta}(\mathcal{O})$ and $u \in \mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$ is a solution of (3.1), then we have*

$$\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} \le N\left(\|\psi f\|_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)} + \|u_0\|_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}\right), \tag{3.11}$$

where $N = N(d,p,\theta,\delta_0,K,\mathcal{O},T)$.

Now we prove a priori estimate for solutions of the equation

$$du = (a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f)\,dt + (\sigma^{ik}u_{x^i} + \mu^k u + g^k)\,dw^k_t, \quad u(0) = u_0. \tag{3.12}$$

Theorem 3.7 *Suppose $\gamma \ge 0$, $\theta \in (d-2+p-\beta_0, d-2+p+\beta_0)$ and Assumption 2.10 are satisfied. Also let $f \in \mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T)$, $g \in \mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)$ and $u_0 \in U^{\gamma+2}_{p,\theta}(\mathcal{O})$. Then estimate (3.11) holds given that $u \in \mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$ is a solution of (3.12).*

Proof. Step 1. Assume

$$|a^{ij}(t,x)-a^{ij}(t,y)|+|\sigma^i(t,x)-\sigma^i(t,y)|_{\ell_2}+|\psi(x)b^i(t,x)|+|\psi^2(x)c(t,x)|+|\psi\mu|_{\ell_2}\le\kappa,\quad \forall\omega,t,x,y.$$

We prove that there exists $\kappa_0=\kappa_0(d,\gamma,\theta,\delta_0,K)>0$ so that the assertion of the theorem holds if $\kappa\le\kappa_0$. Fix $x_0\in\mathcal{O}$ and denote $a_0^{ij}(t,x)=a^{ij}(t,x_0)$ and $\sigma_0^{ik}(t,x)=\sigma^{ik}(t,x_0)$. Then u satisfies

$$du=(a_0^{ij}u_{x^ix^j}+f_0)\,dt+(\sigma_0^{ik}u_{x^i}+g_0^k)\,dw_t^k,\quad u(0)=u_0,$$

where

$$f_0=(a^{ij}-a_0^{ij})u_{x^ix^j}+b^iu_{x^i}+cu+f,\quad g_0^{ik}=(\sigma^{ik}-\sigma_0^{ik})u_{x^i}+\mu^ku+g^k.$$

By Corollary 3.6,

$$\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}\le N\left(\|\psi f_0\|_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)}+\|g_0\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)}+\|u_0\|_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}\right).\tag{3.13}$$

If γ is not integer, then by Lemma 2.2(iii) with some $\nu\in(0,1-\frac{\gamma}{\gamma_+})$ (e.g. $\nu=\frac{1}{2}(1-\frac{\gamma}{\gamma_+})$),

$$\|(a^{ij}-a_0^{ij})\psi u_{x^ix^j}\|_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)}\le N\sup|a^{ij}-a_0^{ij}|^{\nu}\|\psi u_{x^ix^j}\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,T)}\le N\kappa^{\nu}\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\Omega,T)},$$

$$\|\psi b^iu_{x^i}+\psi cu\|_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)}\le N\sup|\psi b^i|^{\nu}\|u_x\|_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)}+N\sup|\psi^2c|\|\psi^{-1}u\|_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)},\le N\kappa^{\nu}\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\Omega,T)},$$

and similarly

$$\|(\sigma^i-\sigma_0^i)u_x\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)}\le N\sup|\sigma^i-\sigma_0^i|^{\nu}_{\ell_2}\|u_x\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)}\le N\kappa^{\nu}\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)},$$

$$\|\mu^ku\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)}\le N\sup|\psi\mu|^{\nu}_{\ell_2}\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)}\le N\kappa^{\nu}\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}.$$

By these and (3.13),

$$\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}\le N\kappa^{\nu}\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}+N\left(\|\psi f\|_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)}+\|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)}+\|u_0\|_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}\right).\tag{3.14}$$

Thus it is enough to take κ_0 so that $N\kappa^{\nu}<1/2$ for all $\kappa\le\kappa_0$.

If $\gamma=0$, then obviously

$$\begin{aligned}&\|\psi(a^{ij}-a_0^{ij})u_{x^ix^j}+\psi b^iu_{x^i}+\psi cu\|_{\mathbb{L}_{p,\theta}(\mathcal{O},T)}\\ \le\;&\sup|a^{ij}-a_0^{ij}|\|\psi u_{xx}\|_{\mathbb{L}_{p,\theta}(\mathcal{O},T)}+\sup|\psi b|\|u_x\|_{\mathbb{L}_{p,\theta}(\mathcal{O},T)}+\sup|\psi^2c|\|\psi^{-1}u\|_{\mathbb{L}_{p,\theta}(\mathcal{O},T)}\\ \le\;&N\kappa\|\psi^{-1}u\|_{\mathbb{H}^{2}_{p,\theta}(\mathcal{O},T)},\end{aligned}$$

and by Lemma 2.2 (also see (2.14)) with $\nu=\varepsilon/(1+\varepsilon)$,

$$\|(\sigma^i-\sigma_0^i)u_x\|_{\mathbb{H}^{1}_{p,\theta}(\mathcal{O},T)}\le N|\sigma^i-\sigma_0^i|^{(0)}_1\|u_x\|_{\mathbb{H}^{1}_{p,\theta}(\mathcal{O},T)}\le N\sup|\sigma^i-\sigma_0^i|^{\nu}\|u_x\|_{\mathbb{H}^{1}_{p,\theta}(\mathcal{O},T)}\le N\kappa^{\nu}\|\psi^{-1}u\|_{\mathbb{H}^{2}_{p,\theta}(\mathcal{O},T)},$$

$$\|\mu u\|_{\mathbb{H}^{1}_{p,\theta}(\mathcal{O},T)}\le N|\psi\mu|^{(0)}_1\|\psi^{-1}\|_{\mathbb{H}^{1}_{p,\theta}(\mathcal{O},T)}\le N\sup|\psi\mu|^{\nu}\|\psi^{-1}u\|_{\mathbb{H}^{1}_{p,\theta}(\mathcal{O},T)}\le N\kappa^{\nu}\|\psi^{-1}u\|_{\mathbb{H}^{2}_{p,\theta}(\mathcal{O},T)}.$$

These lead to (3.14) for $\gamma=0$.

If $\gamma = 1,2,3,...$, then by Lemma 2.2(ii)

$$\begin{aligned}\|(a^{ij}-a_0^{ij})\psi u_{x^ix^j}\|_{\mathbb{H}^\gamma_{p,\theta}(\Omega,T)} &\leq N\sup|a^{ij}-a_0^{ij}|\|\psi u_{x^ix^j}\|_{\mathbb{H}^\gamma_{p,\theta}(\Omega,T)} + N|a^{ij}|^{(0)}_\gamma\|\psi u_{xx}\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T)} \\ &\leq N\kappa\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} + N\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)},\end{aligned}$$

and similarly,

$$\|\psi b^i u_{x^i} + \psi cu\|_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T)} \leq N\kappa\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} + N\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)},$$

$$|(\sigma^i-\sigma^i_0)u_x\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)} \leq N\kappa\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} + N\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)}.$$

Thus if κ_1 is sufficiently small and $\kappa \leq \kappa_1$, then

$$\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} \leq N\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)} + N\left(\|\psi f\|_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)} + \|u_0\|_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}\right).$$

This and the inequality

$$\|\psi^{-1}u\|_{H^{\gamma+1}_{p,\theta}(\mathcal{O})} \leq \varepsilon\|\psi^{-1}u\|_{H^{\gamma+2}_{p,\theta}(\mathcal{O})} + N(\varepsilon)\|\psi^{-1}u\|_{H^2_{p,\theta}(\mathcal{O})},$$

yield

$$\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)} \leq N\|\psi^{-1}u\|_{\mathbb{H}^2_{p,\theta}(\mathcal{O},T)} + N\left(\|\psi f\|_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)} + \|u_0\|_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}\right).$$

Take $\kappa_0 = \kappa_0(0)$ chosen in the above when $\gamma = 0$. Then it suffices to take $\kappa_0 = \kappa_0(\gamma)$ so that $\kappa_0 < \kappa_0(0) \wedge \kappa_1$.

Step 2. We generalize the result of Step 1 by summing up the local estimations of u.

Let $x_0 \in \partial\mathcal{O}$. Fix a nonnegative function $\eta \in C_0^\infty(B_1(0))$ so that $\eta(x) = 1$ for $|x| \leq 1/2$ and define $\eta_n(x) = \eta(n(x-x_0))$,

$$a^{ij}_n(t,x) = a^{ij}(t,x)\eta_n(x) + (1-\eta_n(x))a^{ij}(t,x_0) = a^{ij}(t,x_0) + \eta_n(x)(a^{ij}(t,x)-a^{ij}(t,x_0)),$$

$$\sigma^{ik}_n(t,x) = \eta_n(x)\sigma^{ik}(t,x) + (1-\eta_n(x))\sigma^{ik}(t,x_0),$$

$$b^i_n = b^i\eta_n, \quad c_n = c\eta_n, \quad \mu_n = \eta_n\mu.$$

Then

$$|a^{ij}_n(t,x) - a^{ij}_n(t,y)| \leq 2\sup_{x\in supp\ \eta_n}\eta_n(x)|a^{ij}(t,x)-a^{ij}(t,x_0)|,$$

$$|\sigma^{ij}_n(t,x) - \sigma^{ik}_n(t,y)|_{\ell_2} \leq 2\sup_{x\in supp\ \eta_n}\eta_n(x)|\sigma^{ik}(t,x)-\sigma^{ik}(t,x_0)|_{\ell_2}$$

and for any multi-index α,

$$\sup_n\sup_{x\in\mathcal{O}}\psi^{|\alpha|}|D^\alpha\eta_n| < N(|\alpha|,\eta) < \infty.$$

Indeed, for instance, if x is in the support of η_n, then $\rho(x) \leq 1/n$ and thus $|\rho(x)D\eta_n(x)| = n\rho(x)|\eta_x(n(x-x_0))| \leq \sup_x|\eta_x|$. Using this one can easily check that the coefficients $a^{ij}_n, b^i_n, \cdots, \mu^k_n$ satisfy (2.12), (2.13) and (2.14) with some constant K_0, which is independent of n.

Take κ_0 from Step 1 corresponding to $d, \gamma, \delta_0, K, K_0$ and θ. We fix n large enough so that

$$|a^{ij}_n(t,x) - a^{ij}_n(t,y)| + |\sigma^i_n(t,x) - \sigma^i_n(t,y)|_{\ell_2} + |\psi b^i_n(t,x)| + |\psi^2 c_n(t,x)| + |\psi\mu_n|_{\ell_2} < \kappa_0 \quad \forall \omega, t, x, y.$$

This is possible due to the uniform continuity of a^{ij}, σ^i and condition (2.15).

Now we denote $v = u\eta_{2n}$. Then since $\eta_n = 1$ and e.g. $a^{ij}_n = a^{ij}$ on the support of v, v satisfies

$$dv = (a^{ij}_n v_{x^ix^j} + b^i_n v_{x^i} + c_n v + \bar{f})dt + (\sigma^{ik}_n v_{x^i} + \mu^k_n v + \bar{g}^k_n)dw^k_t, \quad v(0) = u_0\eta_{2n},$$

where

$$\bar{f} := -2a^{ij}u_{x^i}\eta_{2nx^j} - a^{ij}u\eta_{2nx^ix^j} - b^i u\eta_{nx^i} + \eta_{2n}f, \quad \bar{g}^k = -\sigma^{ik}u\eta_{2nx^i} + \eta_{2n}g^k.$$

By the result of Step 1, for each $t \le T$,

$$\begin{aligned}
\|v\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\Omega,t)} &\le N(\|\psi\bar{f}\|^p_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},t)} + \|\bar{g}\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t)} + \|u_0\eta_{2n}\|^p_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}) \\
&\le N\|\psi u_x\|^p_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},t)} + N\|u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t)} + N(\|\psi f\|^p_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},t)} + \|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t,\ell_2)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}) \\
&\le N\|u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t)} + N(\|\psi f\|^p_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},t)} + \|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t,\ell_2)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}),
\end{aligned}$$

where N is independent of t, and the second inequality is due to Lemma 2.2 and the following:

$$|a^{ij}\eta_{2nx}|^{(0)}_{\gamma_+} + |\psi a^{ij}\eta_{2nxx}|^{(0)}_{\gamma_+} + |\psi b^i\eta_{2nx}|^{(0)}_{\gamma_+} + |\sigma^i\eta_{2nx}|^{(0)}_{(\gamma+1)_+} \le N < \infty.$$

Now to estimate u, one introduces a partition of unity $\zeta_i, i = 0, 1, ..., N$ (remember we assume $\mathcal{O}$ is bounded) so that $\zeta_0 \in C^\infty_0(\mathcal{O})$ and $\zeta_i = \eta(2n(x - x_i))$, $x_i \in \partial\mathcal{O}$ for $i \ge 1$. Then by the above result, for each $i \ge 1$ and $t \le T$,

$$\|\zeta_i u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\Omega,t)} \le N(\|u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t)} + \|\psi f\|^p_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},t)} + \|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t,\ell_2)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}). \tag{3.15}$$

Note that since ζ_0 has compact support in $\mathcal{O}$, for any $h \in H^\gamma_{p,\theta}(\mathcal{O})$ we have $\zeta_0 h \in H^\gamma_p$. Moreover for any $\nu \in \mathbb{R}$,

$$\|\psi^\nu\zeta_0 h\|_{H^\gamma_{p,\theta}(\mathcal{O})} \sim \|\psi^\nu\zeta_0 h\|_{H^\gamma_p} \sim \|\zeta_0 h\|_{H^\gamma_p}. \tag{3.16}$$

Write down the equation for $\zeta_0 u$ and apply Theorem 5.1 of [16] to get

$$\begin{aligned}
\|\zeta_0 u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},t)} \sim \|\zeta_0 u\|^p_{\mathcal{H}^{\gamma+2}_p(t)} &\le N\| - 2a^{ij}u_x\zeta_{0x} - a^{ij}u\zeta_{0xx} - b^i u\zeta_{0x} + \zeta_0 f\|^p_{\mathbb{H}^\gamma_p(t)} \\
&\quad + N\|\sigma^i u\zeta_{0x^i} + \zeta_0 g\|^p_{\mathbb{H}^{\gamma+1}_p(t)} + N\|\zeta_0 u_0\|^p_{U^{\gamma+2}_p}.
\end{aligned}$$

Actually the smoothness condition on the coefficients in Theorem 5.1 of [16] is different from ours since there the coefficients are assumed to be in standard Hölder spaces. But since ζ_0 has compact support, one can replace these coefficients with $\bar{a}^{ij}, \bar{b}^i, \cdots, \bar{\mu}^k$ having finite standard Hölder norms without hurting the equation. By (3.16),

$$\|\bar{a}^{ij}u_x\zeta_{0x}\|_{\mathbb{H}^\gamma_p(t)} \le N\|u_x\zeta_{0x}\|_{\mathbb{H}^\gamma_p(t)} \le N\|\psi u_x\zeta_{0x}\|_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},t)} \le N\|\psi u_x\|_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},t)} \le N\|u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t)}.$$

Similar calculus easily shows $\zeta_0 u$ also satisfies (3.15). By summing all these estimates and using (2.10) we get, for $t \leq T$

$$\begin{aligned}
\|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},t)} &\leq N\|u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t)} + N\|\psi f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},t)} + N\|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t)} + N\|u_0\|^p_{U^{\gamma+2}_{p,\theta}(\mathcal{O})} \\
&\leq N\int_0^t \|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},s)} ds + N\left(\|\psi f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)} + \|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)} + \|u_0\|_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}\right).
\end{aligned}$$

Thus estimate (3.11) follows from this and Gronwall's inequality. □

4 Proof of Theorem 2.12

Due to the method of continuity and a priori estimate (3.11) (see e.g. the proof of Theorem 5.1 of [16] for details), to finish the proof, we only show that for any $f \in \psi^{-1}\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T), g \in \mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)$ and $u_0 \in U^{\gamma+2}_{p,\theta}(\mathcal{O})$, the equation

$$du = (\Delta u + f)\,dt + g^k dw^k_t, \quad u(0) = 0 \tag{4.1}$$

has a solution $u \in \mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$. We can approximate $g = (g^1, g^2, ...)$ with functions having only finite nonzero entries, and smooth functions with compact support are dense in $H^{\nu}_{p,\theta}(\mathcal{O})$. Therefore it follows from a priori estimate (3.11) that, to prove existence of solution, we may assume that g has only finite nonzero entries and is bounded on $\Omega \times [0,T] \times \mathcal{O}$ along with each derivative in x and vanishes if x is near $\partial\mathcal{O}$. Indeed, let $g^n \to g$ in $\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)$ where g^n satisfy the above mentioned conditions, and assume that equation (4.1) with g^n in place of g has a solution u^n, then using (3.11) applied for $u^n - u^m$ one easily finds that $\{u^n\}$ is a Cauchy sequence in $\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$ and $u_n \to u$ for some $u \in \mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$. Obviously the limit u becomes a solution of (4.1) (see Theorem 2.7).

Under such assumed conditions on g,

$$v(t,x) := \int_0^t g^k(s,x)dw^k_s$$

is infinitely differentiable in x and vanishes near $\partial\mathcal{O}$. Therefore we conclude $v \in \mathfrak{H}^{\nu}_{p,\theta}(\mathcal{O},T)$ for any $\nu \in \mathbb{R}$. Observe that equation (4.1) can be written as

$$d\bar{u} = (\Delta\bar{u} + f + \Delta v)dt,$$

where $\bar{u} := u - v$. Thus we reduced the case to the case in which $g \equiv 0$. The same argument shows that we may further assume that f, u_0 are bounded along each derivative in (t,x) and vanish near $\partial\mathcal{O}$. Furthermore by considering $u - u_0$, we find that we also may assume $u_0 = 0$.

First, we consider the case $\theta \geq d - 2 + p$.

Lemma 4.1 *Let $\theta \geq d-2+p$, $f \in \mathbb{L}_{p,d}(\mathcal{O},T)$ vanish near $\partial\mathcal{O}$, say $f(t,x)=0$ for $x \notin \mathcal{O}_k := \{x \in \mathcal{O} : \psi(x) > 1/k\}$ for some $k>0$. Also assume that the first derivatives of f in x exist and are bounded. Then the equation*

$$du = (\Delta u + f)\,dt, \quad u(0)=0 \tag{4.2}$$

has a solution $u \in \mathfrak{H}^1_{p,\theta}(\mathcal{O},T)$.

Proof. By Lemma 3.2 we only need to prove that there exists a solution $u \in \psi\mathbb{L}_{p,d-2+p}(\mathcal{O},T)$. Let $n>k$. Since $\partial\mathcal{O}_n \in C^\infty$, by Theorem 2.10 in [11] (c.f. Theorem IV 5.2 in [23]), there is a unique (classical) solution $u^n \in \mathfrak{H}^2_{p,d}(\mathcal{O}_n,T)$ of

$$du^n = (\Delta u^n + f)dt, \quad u^n(0,\cdot)=0,$$

such that $u^n|_{\partial\mathcal{O}_n}=0$ and Du^n, D^2u^n are bounded in $[0,T]\times\mathcal{O}_n$. Extend $u^n(x)=0$ for $x\notin\mathcal{O}_n$, then u^n is Lipschitz continuous in $\mathcal{O}$. Since for any $q\geq 2$, $(|u|^q)_t = q|u|^{q-2}uu_t = q|u|^{q-2}u(\Delta u+f)$, for each $x\in\mathcal{O}_n$,

$$|u^n(T,x)|^q = q\int_0^T |u^n|^{q-2}u^n(\Delta u^n + f)dt.$$

Integrate this over $\mathcal{O}_n$ and do integration by parts to get

$$\begin{aligned}\int_0^T\int_{\mathcal{O}_n} |u^n|^{q-2}|Du^n|^2\,dx\,dt &\leq 1/(q-1)\int_0^T\int_{\mathcal{O}_n}|\psi^{-1}u^n|^{q-1}|\psi f|\psi^{q-2}dx \\ &\leq \varepsilon\int_0^T\int_{\mathcal{O}}|\psi^{-1}u^n|^q\psi^{q-2}\,dx\,dt + N(\varepsilon,q)\int_0^T\int_{\mathcal{O}}|\psi f|^q\psi^{q-2}\,dx\,dt.\end{aligned} \tag{4.3}$$

Taking $q=2$ and using Hardy inequality, we get

$$\sup_n(\|\psi^{-1}u^n\|_{\mathbb{L}_{2,d}(\mathcal{O},T)} + \|Du^n\|_{\mathbb{L}_{2,d}(\mathcal{O},T)}) < \infty.$$

Now we choose $\zeta^n \in C_0^\infty(\mathcal{O}_n)$ such that $\zeta^n=1$ on $\mathcal{O}_k$, $\psi\zeta^n_x, \psi^2\zeta^n_{xx}$ are bounded in $\mathcal{O}$ uniformly in n, and $\zeta^n(x)\to 1$ for $x\in\mathcal{O}$ as $n\to\infty$. Then $u^n\zeta^n \in \mathfrak{H}^2_{2,d}(\mathcal{O},T)$ satisfies

$$(u^n\zeta^n)_t = \Delta(u^n\zeta^n) - 2u^n_{x^i}\zeta^n_{x^i} - u^n\Delta\zeta^n + f.$$

By a priori estimate (3.11)

$$\|u^n\zeta^n\|_{\mathfrak{H}^2_{2,d}(\mathcal{O},T)} \leq N\|u^n_{x^i}\psi\zeta^n_{x^i} - \psi^{-1}u^n\psi^2\Delta\zeta^n\|_{\mathbb{L}_{2,d}(\mathcal{O},T)} + N\|\psi f\|_{\mathbb{L}_{2,d}(\mathcal{O},T)}.$$

By dominated convergence theorem,

$$\|u^n_{x^i}\psi\zeta^n_{x^i} - \psi^{-1}u^n\psi^2\Delta\zeta^n\|_{\mathbb{L}_{2,d}(\mathcal{O},T)} \to 0 \quad \text{as} \quad n\to\infty.$$

Denote $v^n = u^n\zeta^n \in \mathfrak{H}^1_{2,d}(\mathcal{O},T)$, then $\{v^n\}$ is a bounded sequence in $\mathfrak{H}^1_{2,d}(\mathcal{O},T)$. By Theorem 2.7 there exists $u\in\mathfrak{H}^1_{2,d}(T)$ so that v^n and $\mathbb{D}u^n$ converges weakly to u and $\mathbb{D}u$ respectively, and for any $\phi\in C_0^\infty(\mathcal{O})$ and $t\in[0,T]$ we have $(v^n(t),\phi)\to(u(t),\phi)$ weakly in $L_2(\Omega)$. Since $v^n\to u$ weakly in

$\mathbb{H}^1_{2,d-2}(\mathcal{O},T)$, we have $\Delta v^n \to v$ in $\mathbb{H}^{-1}_{2,d+2}(\mathcal{O},T)$. These and the fact that $(-2u^n_{x^i}\zeta^n_{x^i} - u^n\zeta^n_{x^ix^j}, \phi) = 0$ for all large n show that u satisfies (4.2) in the sense of distribution.

Also, (4.3) with $q = p$ and (3.7) certainly show that $\sup_n \|\psi^{-1}u^n\|_{\mathbb{L}_{p,d-2+p}(\mathcal{O},T)} < \infty$. It follows that $\psi^{-1}u \in \mathbb{L}_{p,d-2+p}(\mathcal{O},T) \subset \mathbb{L}_{p,\theta}(\mathcal{O},T)$. The lemma is proved. □

To finish the proof, we only need to show that there exists $\beta_1 > 0$ so that $\theta > d - 2 + p - \beta_1$, then equation (4.2) has a solution $u \in \mathbb{L}_{p,\theta-p}(\mathcal{O},T)$. As before we assume f is sufficiently smooth and vanishes near the boundary. Take κ_0 from Step 1 of the proof of Theorem 3.7. We already proved that if $|\psi b^i| + |\psi^2 c| \leq \kappa_0$ and $\theta = d - 2 + p$, the equation

$$dv = (\Delta v + b^i v_{x^i} + cv + \psi^\beta f), \quad v(0) = 0 \tag{4.4}$$

has a unique solution $v \in \mathfrak{H}^1_{p,\theta}(\mathcal{O},T)$ for any β. Since ψ_x and $\psi\psi_{xx}$ are bounded we can fix $\beta > 0$ so that for

$$b^i := 2\psi^\beta(\psi^{-\beta})_{x^i} = -2\beta\psi^{-1}\psi_{x^i},$$
$$c := \psi^\beta\Delta(\psi^{-\beta}) = \beta(\beta-1)\psi^{-2}|\psi_x|^2 - \beta\psi^{-1}\Delta\psi$$

the inequality $|\psi b^i| + |\psi^2 c| \leq \kappa_0$ holds, and thus (4.4) has a solution $v \in \mathfrak{H}^1_{p,d-2+p}$. Now it is enough to check that $u := \psi^{-\beta}v$ satisfies (4.2) and $u \in \mathfrak{H}^1_{p,d-2+p-\beta p}(\mathcal{O},T) \subset \mathfrak{H}^1_{p,\theta}(\mathcal{O},T)$ for any $\theta \geq d - 2 + p - \beta p$. The theorem is proved.

5 Proof of Theorem 2.15

Our previous proofs (see e.g. Lemma 3.5) show that we only need to consider case $\theta = d$ with equation (3.1) having coefficients independent of x. First observe that inclusion $\mathfrak{H}^{\gamma+2}_{p,d}(\mathcal{O},T) \subset \mathfrak{H}^{\gamma+2}_{p,d-2+p}(\mathcal{O},T)$ gives the uniqueness result for free. Also Remark 3.3 shows that we only need to show there is a solution $u \in \mathbb{L}_{p,d-p}(\mathcal{O},T)$, so that

$$\|\psi^{-1}u\|_{\mathbb{L}_{p,d}(\mathcal{O},T)} \leq N\left(\|\psi f\|_{\mathbb{L}_{p,d}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^1_{p,d}(\mathcal{O},T,\ell_2)} + \|u_0\|_{U^2_{p,d}(\mathcal{O})}\right).$$

For simplicity, assume $u_0 = 0$. Denote

$$\mathcal{F}_{p,\theta} = \{(f,g) : \|(f,g)\|_{\mathcal{F}_{p,\theta}} = \|\psi f\|_{\mathbb{L}_{p,\theta}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^1_{p,\theta}(\mathcal{O},T,\ell_2)} < \infty\}.$$

Fix $q > 2$ and $\beta \in (0,\beta_0)$, where $\beta_0 = \beta_0(d,\delta_0,K)$. Then by Theorem 2.12, the map $\mathcal{R} : (f,g) \to \psi^{-1}u$, where u is the solution of equation (3.1) is a bounded operator from $\mathcal{F}_{2,d-\beta}$ to $\mathbb{L}_{2,d-\beta}(\mathcal{O},T)$, and from $\mathcal{F}_{q,d-2+q}$ to $\mathbb{L}_{q,d-2+q}(\mathcal{O},T)$. Choose $\nu \in (0,1)$ and $p \in (2,q)$ so that $d = (1-\nu)(d-\beta) + \nu(d-2+q)$ and $1/p = (1-\nu)/2 + \nu/q$. Then $F_{p,d}$ (resp. $\mathbb{L}_{p,d}(\mathcal{O},T)$) becomes a complex interpolation space of $F_{2,d-\beta}$ and $F_{q,d-2+q}$ (resp. $\mathbb{L}_{2,d-\beta}(\mathcal{O},T)$ and $\mathbb{L}_{q,d-2+q}(\mathcal{O},T)$), that is,

$$\mathcal{F}_{p,d} = [F_{2,d-\beta}, F_{q,d-2+q}]_\nu, \quad \mathbb{L}_{p,d}(\mathcal{O},T) = [\mathbb{L}_{2,d-\beta}(\mathcal{O},T), \mathbb{L}_{q,d-2+q}(\mathcal{O},T)]_\nu.$$

(See Proposition 2.4 of [27] and Theorem 5.1.2 of [2] for details). It follows from the interpolation theory that $\mathcal{R}$ is a bounded linear map from $\mathcal{F}_{p,d}$ to $\mathbb{L}_{p,d}(\mathcal{O},T)$ (see Theorem (a) on Page 59 of

[34]). This proves the claim for above fixed p. Now for $2 \le p' \le p$, it is enough to notice that for ν' so that $1/p^{'} = (1-\nu')/2 + \nu'/p$,

$$\mathcal{F}_{p',d} = [F_{2,d}, F_{p,d}]_{\nu'}, \quad \mathbb{L}_{p',d}(\mathcal{O},T) = [\mathbb{L}_{2,d}(\mathcal{O},T), \mathbb{L}_{p,d}(\mathcal{O},T)]_{\nu'}.$$

It follows that $\mathcal{R}$ is a bounded linear map from $\mathcal{F}_{p',d}$ to $\mathbb{L}_{p',d}(\mathcal{O},T)$. The theorem is proved.

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